

Constructing quantum games from non-factorizable joint probabilities

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Contents

1	Introduction	2
2	Factorizability and violation of Bell inequality	2
3	Two-player games using factorizable probabilities	4
3.1	Games with coins	4
3.1.1	Two-coin setup	5
3.1.2	Four-coin setup	5
3.2	Examples	7
3.2.1	Prisoner's Dilemma	7
3.2.2	Stag Hunt	8
3.2.3	Chicken game	9
4	Playing games with EPR experiments	10
5	Two-player games using non-factorizable probabilities	11
5.1	Examples	13
5.1.1	Prisoner's Dilemma	13
5.1.2	Stag Hunt	14
5.1.3	Chicken game	17
6	Discussion	18

Abstract

A probabilistic framework is developed that gives a unifying perspective on both the classical and the quantum games. We suggest exploiting peculiar probabilities involved in Einstein-Podolsky-Rosen (EPR) experiments to construct quantum games. In our framework a game attains classical interpretation when joint probabilities are factorizable and a quantum game corresponds when these probabilities cannot be factorized. We analyze how non-factorizability changes Nash equilibria in two-player games while considering the games of Prisoner's Dilemma, Stag Hunt, and Chicken. In this framework we find that for the game of Prisoner's Dilemma even non-factorizable EPR joint probabilities cannot be helpful to escape from the classical outcome of the game. For a particular version of the Chicken game, however, we find that the two non-factorizable sets of joint probabilities, that maximally violates the Clauser-Holt-Shimony-Horne (CHSH) sum of correlations, indeed result in new Nash equilibria.

1 Introduction

Usual approach in the area of quantum games [1, 2, 3, 4, 5, 6, 7, 8, 9] consists of analyzing a quantum system manoeuvred by participating agents, recognized as players, who possess necessary means for their actions on parts of the system. The quantum system evolves to its final state and players' payoffs, or utilities, mathematically expressed as expectation values of self-adjoint payoff operators, are generated from quantum measurement [10]. Thus the usual constructions of quantum games involve the concepts of quantum state vectors, entangled states, quantum measurement, expectation values, trace operation, and that of density operators etc. This may seem normal because as being part of the research field of quantum computation [11] quantum games are expected to exploit relevant tools from quantum mechanics. However, in our experience, this noticeable reliance of the models of quantum games on the tools of quantum mechanics also succeeds to keep many readers away from this inter-disciplinary area of research. Ideally, they would like to see genuine quantum games constructed from elementary probabilistic concepts, as it is the case with many examples in game theory [12]. We find this situation as an opportunity to present a probabilistic approach in which quantum games are constructed without referring to the tools of quantum mechanics.

While looking for the possibility of such an approach, it is encouraging to find that the most unusual character of quantum mechanics can be expressed in terms of probabilities [13] only. For example, Bell inequalities [14, 15, 16, 10] can be written in terms of constraints on joint probabilities relevant to pairs of certain random variables. As probabilities are central to usual analyses in game theory, it seems natural to use the peculiar probabilities, responsible for the violation of Bell inequalities, to construct quantum games. We, therefore, suggest to construct quantum games from the probabilities arising in the Einstein-Podolsky-Rosen (EPR) experiments [17, 14, 16, 18, 10] performed to test the violation of Bell inequalities. The most unusual character of the EPR probabilities, that they may not be factorizable, motivates us, in this paper, to find how non-factorizability can be used to construct quantum games. In other words, we search for the role of non-factorizable probabilities in game-theoretic solution concepts when EPR experiments provide the sets of non-factorizable probabilities.

This explicitly probabilistic approach towards quantum games is expected to be of interest to the readers from such areas as economics [19] and mathematical biology [20], where game theory finds extensive applications and the tools of quantum mechanics are found to be rather alien. Secondly, because of its exclusively probabilistic content, this approach promises to provide a unified perspective for both the classical and the quantum games.

The rest of the paper is organized as follows. Section 2 describes the role of factorizability in deriving Bell inequality. Section 3 discusses playing two-player games using factorizable probabilities and presents two- and four-coin setups to play the well known games of Prisoner's Dilemma (PD), Stag Hunt (SH), and Chicken [12]. Section 4 describes playing two-player games with EPR experiments. Section 5 develops a framework in which factorizable probabilities lead to the classical game whereas non-factorizable probabilities result in the quantum game. Section 6 discusses the results and presents a view for further work.

2 Factorizability and violation of Bell inequality

Factorizability is known to be an interesting property of coupled systems with separated parts, saying [21] that for such systems the probability for a simultaneous pair of outcomes can be expressed as the product of the probability for each outcome separately. Mathematically, this is expressed by writing joint probabilities as arithmetic product of their respective marginals [22]. It turns out to be the most important technical ingredient in the proof of Bell inequality. In fact, J. S. Bell used factorizability in his mathematical formulation of locality [10]. Others [21, 23, 22] have recognized it, for example, by such terms as "conditional stochastic dependence."

We describe factorizability with reference to the standard EPR setup [18, 10] used to derive Bell inequality. This setup consists of two spatially-separated participants, known as Alice and

Bob, who share two-particle systems emitted by the same source. We specify the state of a coupled system by λ . We denote Alice's parameter by a that can be set either at S_1 or at S_2 and denote Bob's parameter by b that can be set either at S'_1 or at S'_2 .

In a run, Alice sets her apparatus either at S_1 or at S_2 and, in either case, on receiving her particle she makes a measurement, the outcome of which is π_A that is either $+1$ or -1 . In the same run, Bob sets his apparatus either at S'_1 or at S'_2 and, in either case, on receiving his particle he makes a measurement, the outcome of which is π_B that can be either $+1$ or -1 . Alice and Bob record the outcomes of their measurements for many runs as they receive two-particle systems emitted from the same source.

We denote the probability that Alice obtains the outcome $\pi_A = +1$ or -1 by $\Pr(\lambda; \pi_A, a)$ and, similarly, we denote the probability that Bob obtains the outcome $\pi_B = +1$ or -1 by $\Pr(\lambda; \pi_B, b)$. Also, we denote the probability that Alice and Bob obtain the outcomes π_A and π_B , respectively, by $\Pr(\lambda; \pi_A, \pi_B, a, b)$. These outcomes result from their choices of the parameters a and b , i.e. which one of the four pairs (S_1, S'_1) , (S_1, S'_2) , (S_2, S'_1) , (S_2, S'_2) is realized in a run.

According to quantum theory, the outcomes π_A and π_B both are completely random and Alice and Bob can only find the probabilities to obtain $+1$ or -1 at the outcomes of their measurements. As it is described in the Section 4, Alice's and Bob's parameters a and b decide these probabilities.

In many runs, Alice can choose between S_1 or S_2 with some probability. Similarly, in many runs, Bob can choose between S'_1 or S'_2 with some probability.

Assume that the source emits a total of N two-particle systems. We denote by $N(\pi_A; a)$ the number of times Alice gets the outcome π_A when she may set her parameter a either at S_1 or at S_2 . Similarly, we denote by $N(\pi_B; b)$ the number of times Bob gets the outcome π_B when he may set his parameter b either at S'_1 or at S'_2 . And, we denote by $N(\pi_A, \pi_B; a, b)$ the number of times when Alice gets the outcome π_A and Bob gets the outcome π_B , wherever they may set their parameters a and b , respectively. When N is large, the ensemble probabilities are defined as

$$\begin{aligned} \Pr(\pi_A; a) &= N(\pi_A; a)/N, & \Pr(\pi_A; b) &= N(\pi_B; b)/N, \\ \Pr(\pi_A, \pi_B; a, b) &= N(\pi_A, \pi_B; a, b)/N. \end{aligned} \quad (1)$$

For many runs we consider an ensemble of states emitted from the source that may not be the same. To allow mixture of states we denote the normalized probability density, characterizing the ensemble of emissions, by $\rho(\lambda)$. The ensemble probabilities are

$$\begin{aligned} \Pr(\pi_A; a) &= \int_{\Gamma} \rho(\lambda) \Pr(\lambda; \pi_A; a) d\lambda, \\ \Pr(\pi_B; b) &= \int_{\Gamma} \rho(\lambda) \Pr(\lambda; \pi_B; b) d\lambda, \\ \Pr(\pi_A, \pi_B; a, b) &= \int_{\Gamma} \rho(\lambda) \Pr(\lambda; \pi_A, \pi_B; a, b) d\lambda, \end{aligned} \quad (2)$$

where Γ is the space of states λ . Now, factorizability states that

$$\Pr(\lambda; \pi_A, \pi_B; a, b) = \Pr(\lambda; \pi_A; a) \Pr(\lambda; \pi_B; b). \quad (3)$$

That is, for each λ the joint probabilities are arithmetic product of their respective marginals.

Factorizability being the most important technical ingredient in the proof of Bell inequality can be seen as follows. Following inequalities must hold when probabilities are sensible quantities

$$0 \leq \Pr(\lambda; \pi_A; a) \leq 1, \quad 0 \leq \Pr(\lambda; \pi_B; b) \leq 1. \quad (4)$$

We now refer to a theorem stating that if six numbers $\omega_1, \omega_2, \varpi_1, \varpi_2, \varsigma_1, \varsigma_2$ are given such that

$$0 \leq \omega_1 \leq \varsigma_1, \quad 0 \leq \omega_2 \leq \varsigma_1, \quad 0 \leq \varpi_1 \leq \varsigma_2, \quad 0 \leq \varpi_2 \leq \varsigma_2, \quad (5)$$

then the function $\Xi = \omega_1 \varpi_1 - \omega_1 \varpi_2 + \omega_2 \varpi_1 + \omega_2 \varpi_2 - \varsigma_2 \omega_2 - \varsigma_1 \varpi_1$ is constrained by the inequalities

$$-\varsigma_1 \varsigma_2 \leq \Xi \leq 0. \quad (6)$$

Proof of this theorem can, for example, be found in Ref. [25]. Inequalities (4) together with the inequalities (6) give

$$\begin{aligned}
-1 \leq & \Pr(\lambda; \pi_A; S_1) \Pr(\lambda; \pi_B; S'_1) - \Pr(\lambda; \pi_A; S_1) \Pr(\lambda; \pi_B; S'_2) + \\
& \Pr(\lambda; \pi_A; S_2) \Pr(\lambda; \pi_B; S'_1) + \Pr(\lambda; \pi_A; S_2) \Pr(\lambda; \pi_B; S'_2) - \\
& \Pr(\lambda; \pi_B; S_2) - \Pr(\lambda; \pi_B; S'_1) \leq 0,
\end{aligned} \tag{7}$$

for each λ . The central role of factorizability follows as when the definition (3) of factorizability holds, the multiplication of (7) by $\rho(\lambda)$ and the subsequent integration over λ gives

$$\begin{aligned}
-1 \leq & \Pr(\pi_A, \pi_B; S_1, S'_1) - \Pr(\pi_A, \pi_B; S_1, S'_2) + \Pr(\pi_A, \pi_B; S_2, S'_1) + \\
& \Pr(\pi_A, \pi_B; S_2, S'_2) - \Pr(\pi_A; S_2) - \Pr(\pi_B; S'_1) \leq 0,
\end{aligned} \tag{8}$$

which is the Bell (or Clauser-Horne) inequality [25]. Note that the definition (3) is crucial in order to obtain the inequality (8).

3 Two-player games using factorizable probabilities

To bring non-factorizability into the realm of two-player games, we consider a symmetric two-player, two-strategy, non-cooperative game [12] represented by the matrices

$$\mathcal{A} = \begin{array}{c} \text{Alice} \\ \begin{array}{cc} X_1 & X_2 \end{array} \end{array} \begin{array}{c} \text{Bob} \\ \begin{array}{cc} X'_1 & X'_2 \end{array} \end{array} \begin{pmatrix} K & L \\ M & N \end{pmatrix}, \quad \mathcal{B} = \begin{array}{c} \text{Alice} \\ \begin{array}{cc} X_1 & X_2 \end{array} \end{array} \begin{array}{c} \text{Bob} \\ \begin{array}{cc} X'_1 & X'_2 \end{array} \end{array} \begin{pmatrix} K & M \\ L & N \end{pmatrix}, \tag{9}$$

where all K, L, M, N are real numbers. Players can go for one of the two available strategies: X_1, X_2 for Alice and X'_1, X'_2 for Bob.

As factorizability is central to obtain Bell inequality, in this paper we construct quantum games from non-factorizable probabilities that exploit EPR setup. This rests on Fine's view [21] that the violation of Bell inequality in EPR experiments shows that quantum theory violates factorizability. This view allows us to construct quantum games for which factorizability always corresponds to the classical game.

We recognize key features of an EPR setup being that these relate to a probabilistic system divided into two parts such that a) each observer has access to one part of the system b) each observer can select between two available choices c) observers cannot communicate between themselves d) observers can make independent selections between the available choices e) probabilities relevant to each part of the system are normalized¹ and that f) probabilities are sensible quantities.

It is worth mentioning here that the experimental testing of Bell inequality involves four correlation experiments that correspond to combining S_1 with S'_1 , S_1 with S'_2 , S_2 with S'_1 , and S_2 with S'_2 , respectively. These experiments are mutually exclusive in the sense that for any given experiment Alice has to select between S_1 and S_2 and Bob has to select between S'_1 and S'_2 . That is, Alice (Bob) cannot go for S_1 (S'_1) and S_2 (S'_2) simultaneously because the corresponding observables are incompatible, and cannot be measured simultaneously. Whereas, in the above derivation of the Bell inequality it is assumed that S_1, S'_1, S_2, S'_2 all have definite values which can be measured simultaneously in pairs.

3.1 Games with coins

The above mentioned features are remindful of coins which, if distributed between players, are found to have all the above mentioned properties. For coins factorizability has a straightforward meaning in that the associated probabilities remain factorizable. Hence, we develop an analysis of two-player games with non-factorizable probabilities by first translating playing of three well known games in terms of the games played when players share coins. It turns out that a version

¹Its exact meaning will be described shortly.

of this translation provides the right comparison with the probabilities involved in the EPR experiments and opens the way to the next step i.e. to introduce non-factorizable probabilities into the playing of two-player games.

3.1.1 Two-coin setup

We now consider pairs of coins and use it to play a two-player game (9). For example, this game can be played when each player receives a coin, head up, and ‘to flip’ or to ‘not to flip’ is a player’s strategy. Both coins are then passed to a referee who rewards the players after observing the state of both coins.

Assume S_1 (to flip) and S_2 (not to flip) are Alice’s strategies and S'_1 (to flip) and S'_2 (not to flip) are Bob’s strategies. That is, with reference to the matrices (9), we make the association $S_1 \sim X_1$, $S_2 \sim X_2$, and $S'_1 \sim X'_1$, $S'_2 \sim X'_2$. In two-coin setup, we assume that the strategies S_1 and S'_1 represent Alice’s and Bob’s actions ‘to flip’ the coin, respectively; and, similarly, S_2 and S'_2 represent Alice’s and Bob’s actions ‘not to flip’ the coin, respectively.

In repeated runs of the game players can play mixed strategies. Alice’s mixed strategy $x \in [0, 1]$ is the probability to choose S_1 over S_2 and similarly Bob’s mixed strategy $y \in [0, 1]$ is the probability to choose S'_1 over S'_2 . Players’ payoffs are written as

$$\Pi_{A,B}(x, y) = \begin{pmatrix} x \\ 1-x \end{pmatrix}^T \begin{pmatrix} (K, K) & (L, M) \\ (M, L) & (N, N) \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix}, \quad (10)$$

where T is for transpose and the subscripts A and B refer to Alice and Bob, respectively. The first and the second entries in a parentheses are Alice’s and Bob’s payoffs, respectively. Assume (x^*, y^*) is a Nash equilibrium (NE) [12]:

$$\Pi_A(x^*, y^*) - \Pi_A(x, y^*) \geq 0, \quad \Pi_B(x^*, y^*) - \Pi_B(x^*, y) \geq 0. \quad (11)$$

In rest of this paper we will use “NE” when we refer to either a Nash equilibrium or to Nash equilibria, assuming that the right meaning can be judged from the context. We identify this arrangement to play a two-player game using two coins as the *two-coin setup*.

3.1.2 Four-coin setup

The game (9) can also be played using four coins instead of two. It is arranged by assigning two coins to each player before the game is played. In a run each player has to choose one coin. Two coins out of four are, therefore, chosen by the players in each turn. These coins are then passed to a referee who tosses them together and observes the outcome. It is assumed that the players do not need to share fair coins.

We recall that in two-coin setup S_1 and S'_1 are Alice’s and Bob’s strategies, respectively, that represent players’ actions ‘to flip’ the coin that a player receives in a turn. Instead of flipping or not flipping, in four-coin setup a player’s strategy is to choose one out of the two coins that are made available to each player in a turn. The four-coin setup is relevant as, in a run, choosing a coin out of the two corresponds to choosing one of the two directions in which measurement is performed in standard EPR experiment, the outcome of which is $+1$ or -1 .

In repeated game, a player’s strategy is defined by the selection she/he makes over several runs of the game. For example, a player plays a pure strategy when she/he goes for the same coin over all the runs and plays a mixed strategy when she/he finds a probability to choose one coin out of the two over many runs. Referee rewards the players according to their strategies, the underlying statistics of four coins obtained from the outcomes of many tosses each one of which follows every time the two players choose two out of the total four coins, and the matrices (9) representing the game being played.

We identify the arrangement using four coins to play a two-player game as the *four-coin setup*. Note that in four-coin setup the players’ rewards depend on the outcomes of repeated tosses even for pure strategies. A large number of runs are, therefore, necessary whether a player plays a

pure-strategy or a mixed-strategy. Four-coin setup provides an inherently probabilistic character to playing a two-player game and facilitates a probabilistic analysis when we seek to play the game (9) using EPR experiments.

As the four-coin setup uses a different definition of a strategy relative to the two-coin setup, we call S_1 and S_2 being Alice's coins and S'_1 and S'_2 being Bob's coins. When selecting a coin is a player's strategy and we want to play the game given by the matrices (9), it is reasonable to make the association $S_1 \sim X_1$, $S_2 \sim X_2$, and $S'_1 \sim X'_1$, $S'_2 \sim X'_2$.

We represent the head of a coin by +1 and its tail by -1 and adapt this convention in the rest of this paper. For coins, Alice's outcome of $\pi_A = +1$ or -1 (whether she goes for S_1 or S_2) is independent from Bob's outcome of $\pi_B = +1$ or -1 (whether he goes for S'_1 or S'_2) and relevant joint probabilities are factorizable.

Referring to the definition (3) of factorizability and noticing that probabilities associated to coins are factorizable, we use the same notation that is introduced in Section 2 to consider, for example, the probability $\Pr(\pi_A, \pi_B; S_1, S'_1)$ that can be factorized as $\Pr(\pi_A; S_1) \Pr(\pi_B; S'_1)$.

We define probabilities $r, r' \in [0, 1]$ by $r = \Pr(+1; S_1)$ and $r' = \Pr(+1; S'_1)$ saying that r is the probability of getting head for Alice's first coin S_1 and r' is the probability of getting head for Bob's first coin S'_1 . Factorizability then allows us to write $\Pr(+1, -1; S_1, S'_1) = r(1 - r')$ and $\Pr(-1, -1; S_2, S'_2) = (1 - s)(1 - s')$ where $s = \Pr(+1; S_2)$ and $s' = \Pr(+1; S'_2)$ i.e. s and s' are the probabilities of getting head for Alice's and Bob's second coin, respectively.

In four-coin setup we find it useful to have the following table:

		Bob					
		S'_1		S'_2			
		+1	-1	+1	-1		
Alice	S_1	+1	rr'	$r(1-r')$	rs'	$r(1-s')$	(12)
		-1	$r'(1-r)$	$(1-r)(1-r')$	$s'(1-r)$	$(1-r)(1-s')$	
	S_2	+1	sr'	$s(1-r')$	ss'	$s(1-s')$	
		-1	$r'(1-s)$	$(1-s)(1-r')$	$s'(1-s)$	$(1-s)(1-s')$	

from which we define payoff relations for the players:

$$\begin{aligned} \Pi_{A,B}(S_1, S'_1) &= \mathfrak{r}^T(\mathcal{A}, \mathcal{B}) \mathfrak{r}', & \Pi_{A,B}(S_1, S'_2) &= \mathfrak{r}^T(\mathcal{A}, \mathcal{B}) \mathfrak{s}', \\ \Pi_{A,B}(S_2, S'_1) &= \mathfrak{s}^T(\mathcal{A}, \mathcal{B}) \mathfrak{r}', & \Pi_{A,B}(S_2, S'_2) &= \mathfrak{s}^T(\mathcal{A}, \mathcal{B}) \mathfrak{s}', \end{aligned} \quad (13)$$

where

$$\mathfrak{r} = \begin{pmatrix} r \\ 1 - r \end{pmatrix}, \quad \mathfrak{s} = \begin{pmatrix} s \\ 1 - s \end{pmatrix}, \quad \mathfrak{r}' = \begin{pmatrix} r' \\ 1 - r' \end{pmatrix}, \quad \mathfrak{s}' = \begin{pmatrix} s' \\ 1 - s' \end{pmatrix}. \quad (14)$$

For example, $\Pi_A(S_1, S'_2)$ is Alice's payoff when, in repeated runs of coin tossing, she always goes for her first coin, i.e. S_1 , while Bob goes for his second coin, i.e. S'_2 .

As it is the case with two-coin setup, Alice's mixed strategy in four-coin setup is the probability with which she chooses her pure strategy² S_1 over her other pure strategy S_2 during repeated runs of the experiment. Similarly, Bob's mixed strategy is the probability with which he chooses his pure strategy S'_1 over his other pure strategy S'_2 during repeated runs of the experiment. Assume that Alice plays S_1 with probability x and Bob plays S'_1 with probability y , their mixed-strategy payoff relations are

$$\Pi_{A,B}(x, y) = \begin{pmatrix} x \\ 1 - x \end{pmatrix}^T \begin{pmatrix} \Pi_{A,B}(S_1, S'_1) & \Pi_{A,B}(S_1, S'_2) \\ \Pi_{A,B}(S_2, S'_1) & \Pi_{A,B}(S_2, S'_2) \end{pmatrix} \begin{pmatrix} y \\ 1 - y \end{pmatrix}. \quad (15)$$

The NE can then be found from (11), which is written as

²Notice that our definition of a pure strategy corresponds to the usual mixed-strategy. This agrees with the result in quantum games that a product pure state corresponds to a mixed-strategy classical game.

$$\begin{aligned} (\mathfrak{x} - \mathfrak{s})^T \mathcal{A} \{y^*(\mathfrak{x}' - \mathfrak{s}') + \mathfrak{s}'\} (x^* - x) &\geq 0, \\ \{x^*(\mathfrak{x} - \mathfrak{s})^T + \mathfrak{s}^T\} \mathcal{B}(\mathfrak{x}' - \mathfrak{s}')(y^* - y) &\geq 0. \end{aligned} \quad (16)$$

In the following, before we make a transition to playing our game using EPR experiments, we consider playing three well known games using both the two- and the four-coin setups.

3.2 Examples

We analyze the games of PD, SH, and Chicken in two- and four-coin setups and afterwards make a transition to the EPR setup. PD is known to be a representative of the problems of social cooperation [12] and has been one of the earliest [2] and favorite topics for quantum games. Hence it is worthwhile to analyze this game in the setup using non-factorizable probabilities. Our second game is SH that, like PD, describes conflict between safety and social cooperation. Our third game is Chicken, also known as the Hawk-Dove game [12], which is considered an influential model of conflict for two players in game theory.

3.2.1 Prisoner's Dilemma

PD is a noncooperative game [12] that is widely known to economists, social and political scientists, and in recent years to quantum physicists. It is one of the earliest games to be investigated in the quantum regime [2]. Its name comes from the following situation: two criminals are arrested after having committed a crime together. Each suspect is placed in a separate cell and may choose between two strategies: *to confess* (D) and *not to confess* (C), where C and D stand for Cooperation and Defection.

If neither suspect confesses, i.e. (C, C) , they go free, which is represented by K units of payoff for each suspect. When one prisoner confesses (D) and the other does not (C), the prisoner who confesses gets M units of payoff, which represents freedom as well as financial reward i.e. $M > K$, while the prisoner who did not confess gets L , represented by his ending up in the prison. When both prisoners confess, i.e. (D, D) , both are given a reduced term represented by N units of payoff, where $N > L$, but it is not so good as going free i.e. $K > N$.

Referring to the matrices (9) we make the association $X_1, X'_1 \sim C$ and $X_2, X'_2 \sim D$ and require that $M > K > N > L$. We define $\Delta_1 = (M - K)$, $\Delta_2 = (N - L)$, $\Delta_3 = (\Delta_2 - \Delta_1)$ which requires $\Delta_1, \Delta_2 > 0$ for this game. In two-coin setup, the inequalities (11) give

$$\begin{aligned} \Pi_A(x^*, y^*) - \Pi_A(x, y^*) &= (y^* \Delta_3 - \Delta_2)(x^* - x) \geq 0, \\ \Pi_B(x^*, y^*) - \Pi_B(x^*, y) &= (x^* \Delta_3 - \Delta_2)(y^* - y) \geq 0, \end{aligned} \quad (17)$$

and $(x^*, y^*) = (0, 0)$ comes out as a unique NE at which players' payoffs are $\Pi_A(S_1, S'_1) = N = \Pi_B(S_1, S'_1)$.

In the four-coin setup, the PD game as defined above is played as follows. Using the mixed-strategy payoff relation (15), the pair of pure strategies (S_2, S'_2) is represented by $(x^*, y^*) = (0, 0)$. If we require this strategy pair to be a NE then we also need to know about the constraints this requirement imposes on r, s, r', s' . When $(x^*, y^*) = (0, 0)$ the NE inequalities (16) for PD are reduced to

$$\begin{aligned} -x(s - r)\Delta_2 \{(\Delta_1/\Delta_2 - 1)s + 1\} &\geq 0, \\ -y(s' - r')\Delta_2 \{(\Delta_1/\Delta_2 - 1)s' + 1\} &\geq 0. \end{aligned} \quad (18)$$

Now, for the NE inequalities (18) to hold, it is required that $(s - r) \leq 0$ and $(s' - r') \leq 0$ both when $\Delta_1/\Delta_2 \geq 1$ and when $\Delta_1/\Delta_2 < 1$. This, of course, is possible if

$$\Pr\{S_2(+1)\} = s = 0 = s' = \Pr\{S'_2(+1)\}, \quad (19)$$

which must hold if the strategy pair $(x^*, y^*) = (0, 0)$ is to be a NE in PD. Eqs. (19) should be true along with that the probabilities $\Pr(\pi_A, \pi_B; a, b)$ are factorizable. As we find it, this result

provides the basis on which the forthcoming argument for the quantum version of this game rests. Notice that, from (13), we obtain $\Pi_A(S_2, S'_2) = \Pi_B(S_2, S'_2) = N$ when Eq. (19) holds.

The constraint (19) appears when the strategy pair (S_2, S'_2) is assumed to be the NE. One can assume other strategy pair, for example (S_1, S'_1) , to be a NE for which, instead of the requirement (19), we obtain

$$\Pr\{S_1(+1)\} = r = 0 = r' = \Pr\{S'_1(+1)\}. \quad (20)$$

However, it is found that this freedom does not affect the forthcoming argument for a quantum game.

3.2.2 Stag Hunt

Along with PD, the game of SH provides another interesting context to study problems of social cooperation. It describes the situation when two hunters can either jointly hunt a stag (an adult deer that makes a large meal) or individually hunt a rabbit (which is tasty but makes substantially small meal). Hunting a stag is quite challenging and hunters need to cooperate with each other, especially, it is quite unlikely that a hunter hunts a stag alone.

It is found that, in contrast to PD that has a single pure NE, the game of SH has three NE, two of which are pure and one is mixed. The two pure NE correspond to the situations when both hunters hunt the stag as a team and when each hunts rabbit by himself. SH differs from PD in that mutual cooperation gives maximum reward to the hunters. When compared to PD, SH is considered a better model for the problems of (social) cooperation.

Referring to the matrices (9) the game of SH is defined by

$$K > M \geq N > L \text{ and } M + N > K + L. \quad (21)$$

In two-coin setup the NE inequalities for this game remain the same as the inequalities (17) except that now we have

$$\Delta_3 > \Delta_2 > 0 \text{ and } 0 > \Delta_1, \quad (22)$$

instead of the condition $\Delta_1, \Delta_2 > 0$ that hold for PD. Here $\Delta_1, \Delta_2, \Delta_3$ are defined in the Section 3.2.1. This leads to three NE:

$$\begin{aligned} (x^*, y^*)_1 &= (0, 0); \\ (x^*, y^*)_2 &= (\Delta_2/\Delta_3, \Delta_2/\Delta_3); \\ (x^*, y^*)_3 &= (1, 1); \end{aligned} \quad (23)$$

and the corresponding payoffs at these equilibria, obtained from Eqs. (10), are

$$\begin{aligned} \Pi_A(x^*, y^*)_1 &= N = \Pi_B(x^*, y^*)_1; \\ \Pi_A(x^*, y^*)_2 &= (\Delta_2/\Delta_3)^2 \Delta_3 + (\Delta_2/\Delta_3) \Delta_4 + N = \Pi_B(x^*, y^*)_2; \\ \Pi_A(x^*, y^*)_3 &= K = \Pi_B(x^*, y^*)_3; \end{aligned} \quad (24)$$

where we define $\Delta_4 = L + M - 2N$.

Now consider playing this game within the four-coin setup in which the NE inequalities (16) reduce to

$$\begin{aligned} (r - s)[y^*(r' - s')\Delta_3 + (s'\Delta_3 - \Delta_2)](x^* - x) &\geq 0, \\ (r' - s')[x^*(r - s)\Delta_3 + (s\Delta_3 - \Delta_2)](y^* - y) &\geq 0. \end{aligned} \quad (25)$$

From the inequalities (25) the NE $(x^*, y^*)_1 = (0, 0)$ results when

$$s = 0 = s' \quad (26)$$

and, similarly, the NE $(x^*, y^*)_3 = (1, 1)$ results when

$$r = 0 = r'. \quad (27)$$

Also, the inequalities (25) hold when $x^* = (s\Delta_3 - \Delta_2)/(s - r)\Delta_3$, $y^* = (s'\Delta_3 - \Delta_2)/(s' - r')\Delta_3$ and for $(x^*, y^*)_2 = (\Delta_2/\Delta_3, \Delta_2/\Delta_3)$ to be a NE we require

$$s = 0, r = 1 \text{ and } s' = 0, r' = 1. \quad (28)$$

These constraints on r, s, r', s' hold along with the probabilities $\Pr(\pi_A, \pi_B; a, b)$ being factorizable.

3.2.3 Chicken game

The game of Chicken is about two drivers who drive towards each other from opposite directions. One driver must turn aside, or both may die in a crash. If one driver turns aside but the other does not, s/he will be called a “chicken”. While each driver prefers not to yield to the opponent, the outcome where neither driver yields is the worst possible one for both. In this anti-coordination game it is mutually beneficial for parties to play different strategies.

Sometimes, Chicken is also known as the “Hawk-Dove” game, that originates from the parallel development of the basic principles of this game in two different research areas: economics and mathematical biology. Economists, and the political scientists too, refer to [19] this game as Chicken, while mathematical biologists refer to [20] it as the Hawk-Dove game.

The game of Chicken differs from PD in that in Chicken the mutual defection (the crash when both players drive straight) is the most feared outcome. While in PD cooperation while the other player defects is the worst outcome.

A version of the Chicken game is obtained from the matrices (9) when

$$K = 0, L = \alpha, M = \beta, N = 0, 0 < \alpha < (\alpha + \beta). \quad (29)$$

Playing this game in two-coin setup the inequalities (11) are reduced to

$$\{\alpha - y^*(\alpha + \beta)\}(x^* - x) \geq 0, \quad \{\alpha - x^*(\alpha + \beta)\}(y^* - y) \geq 0, \quad (30)$$

and three NE emerge:

$$\begin{aligned} (x^*, y^*)_1 &= (1, 0); \\ (x^*, y^*)_2 &= (\alpha/(\alpha + \beta), \alpha/(\alpha + \beta)); \\ (x^*, y^*)_3 &= (0, 1). \end{aligned} \quad (31)$$

The corresponding payoffs at these equilibria, obtained from Eqs. (10), are

$$\begin{aligned} \Pi_A(x^*, y^*)_1 &= \alpha, \quad \Pi_B(x^*, y^*)_1 = \beta; \\ \Pi_A(x^*, y^*)_2 &= \alpha\beta/(\alpha + \beta) = \Pi_B(x^*, y^*)_2; \\ \Pi_A(x^*, y^*)_3 &= \beta, \quad \Pi_B(x^*, y^*)_3 = \alpha. \end{aligned} \quad (32)$$

Now we play this game using the four-coin setup. The NE inequalities come out to be the same as the ones given in (25) except that now we have $\Delta_3 = -(\alpha + \beta)$ and $\Delta_2 = -\alpha$. Then for $(x^*, y^*)_1 = (1, 0)$ we require

$$r = 0 \text{ and } s' = 0. \quad (33)$$

Similarly, for $(x^*, y^*)_3 = (0, 1)$ we require

$$r' = 0 \text{ and } s = 0. \quad (34)$$

At $(x^*, y^*)_2 = (\alpha/(\alpha + \beta), \alpha/(\alpha + \beta))$ the inequalities (25) reduce to

$$(r - s)(\alpha - \alpha r' - \beta s')(\alpha/(\alpha + \beta) - x) \geq 0, \quad (r' - s')(\alpha - \alpha r - \beta s)(\alpha/(\alpha + \beta) - y) \geq 0, \quad (35)$$

which puts constraint on r, s, r', s' given as

$$\alpha(1 - r') = \beta s', \quad \alpha(1 - r) = \beta s. \quad (36)$$

A special case is the one when $\alpha = \beta$ and the strategy pair $(x^*, y^*) = (1/2, 1/2)$ becomes a NE which imposes certain constraints on r, s, r', s' . For this NE the inequalities (16), for the game defined by (9, 29), are reduced to

$$\begin{aligned} (r - s) \{ -(\alpha + \beta)(r' + s')/2 + L \} (1/2 - x) &\geq 0, \\ (r' - s') \{ -(\alpha + \beta)(r + s)/2 + L \} (1/2 - y) &\geq 0, \end{aligned} \quad (37)$$

so, we require

$$r + s = 1 = r' + s', \quad (38)$$

if the strategy pair $(x^*, y^*) = (1/2, 1/2)$ is to be a NE in this game. Along with this, the probabilities $\Pr(\pi_A, \pi_B; a, b)$ are to be factorizable.

4 Playing games with EPR experiments

Section 3 describes playing a two-player game with four coins such that choosing a coin is a strategy while players' payoffs are given by their strategies, the matrix of the game, and the underlying statistics of the coins. This facilitates transition to playing the *same* game using EPR experiments.

In EPR setup, Alice and Bob are spatially separated and are unable to communicate with each other. In an individual run, both receive one half of a pair of particles originating from a common source. In the same run of the experiment both choose one from two given (pure) strategies. These strategies are the two directions in space along which spin or polarization measurements can be made.

Keeping the notation for the coins, we denote these directions to be S_1, S_2 for Alice and S'_1, S'_2 for Bob. Each measurement generates +1 or -1 as the outcome, like it is the case with coins after their toss in the four-coin setup. Experimental results are recorded for a large number of individual runs of the experiment and payoffs are awarded depending on the directions the players go for over many runs (defining their strategies), the matrix of the game they play, and the statistics of the measurement outcomes.

For EPR experiments, we retain Cereceda' notation [26] for the associated probabilities:

$$p_k = \Pr(\pi_A, \pi_B; a, b) \quad \text{with} \quad k = 1 + \frac{(1 - \pi_B)}{2} + 2\frac{(1 - \pi_A)}{2} + 4(b - 1) + 8(a - 1). \quad (39)$$

In this notation, for example, we write p_1 for the probability $\Pr(+1, +1; S_1, S'_1)$ and p_8 for the probability $\Pr(-1, -1; S_1, S'_2)$. One can then construct the following table of probabilities

		Bob				
		S'_1		S'_2		
		+1	-1	+1	-1	
Alice	S_1	+1	p_1	p_2	p_5	p_6
		-1	p_3	p_4	p_7	p_8
	S_2	+1	p_9	p_{10}	p_{13}	p_{14}
		-1	p_{11}	p_{12}	p_{15}	p_{16}

This table allows to transparently see how the probabilities $p_i (1 \leq i \leq 16)$ are linked to the probabilities $\Pr(\pi_A, \pi_B; a, b)$, where we recall that a can be set at S_1 or at S_2 and, similarly, b can be set at S_2 or at S'_2 . In Cereceda's notation the EPR probabilities p_i are normalized as they satisfy the following relations

$$\begin{aligned}
p_1 + p_2 + p_3 + p_4 &= 1, \\
p_5 + p_6 + p_7 + p_8 &= 1, \\
p_9 + p_{10} + p_{11} + p_{12} &= 1, \\
p_{13} + p_{14} + p_{15} + p_{16} &= 1.
\end{aligned} \tag{41}$$

Notice that the factorizable probabilities (12) are also normalized and (41) holds for them.

Payoff relations (13) are originally constructed when the game given by the matrices (9) is played with four coins and their mathematical form convinces one to use the following recipe [24] to reward players when the same game is played using EPR probabilities (40):

$$\begin{aligned}
\Pi_A(S_1, S'_1) &= Kp_1 + Lp_2 + Mp_3 + Np_4, \\
\Pi_A(S_1, S'_2) &= Kp_5 + Lp_6 + Mp_7 + Np_8, \\
\Pi_A(S_2, S'_1) &= Kp_9 + Lp_{10} + Mp_{11} + Np_{12}, \\
\Pi_A(S_2, S'_2) &= Kp_{13} + Lp_{14} + Mp_{15} + Np_{16}.
\end{aligned} \tag{42}$$

Here $\Pi_A(S_1, S'_2)$, for example, is Alice's payoff when she plays S_1 and Bob plays S'_2 . Like it is the case with four coins, the payoff relations for Bob are obtained from (42) by the transformation $L \leftrightarrow M$ in Eqs. (42).

When p_i are factorizable in terms of r, r', s, s' , a comparison of (42) with (13) requires

$$\begin{aligned}
p_1 &= rr', & p_2 &= r(1-r'), & p_3 &= r'(1-r), & p_4 &= (1-r)(1-r'), \\
p_5 &= rs', & p_6 &= r(1-s'), & p_7 &= s'(1-r), & p_8 &= (1-r)(1-s'), \\
p_9 &= sr', & p_{10} &= s(1-r'), & p_{11} &= r'(1-s), & p_{12} &= (1-s)(1-r'), \\
p_{13} &= ss', & p_{14} &= s(1-s'), & p_{15} &= s'(1-s), & p_{16} &= (1-s)(1-s').
\end{aligned} \tag{43}$$

That is, the factorizability of p_i in terms r, r', s, s' makes the game played by EPR probabilities equivalent to the one played by using coins.

However, the EPR probabilities p_i , appearing in (13), may not be factorizable in terms of r, s, r', s' , whereas for both the payoff relations (13, 42) the normalization (41) continues to hold.

5 Two-player games using non-factorizable probabilities

As it is the case with the coin game, Alice's mixed strategy is defined to be the probability to choose between S_1 and S_2 and we can use, once again, the payoff relations (15) which, however, now correspond to the possible situation when p_i may not be factorizable. So that, the relations (13) can be replaced with the relations (42) in Alice's mixed-strategy payoff relation in (15). The same applies to Bob's payoff relations.

Note that when p_i are factorizable, using (43) allows the probabilities r, r', s, s' to be expressed in terms of p_i :

$$r = p_1 + p_2, \quad s = p_9 + p_{10}, \quad r' = p_1 + p_3, \quad s' = p_5 + p_7, \tag{44}$$

which are useful relations for the forthcoming argument for a quantum game.

Along with the normalization (41), the EPR probabilities p_i ($1 \leq i \leq 16$) also satisfy certain other constraints imposed by the requirements of causality. Cereceda [26] writes these constraints as

$$\begin{aligned}
p_1 + p_2 - p_5 - p_6 &= 0, & p_1 + p_3 - p_9 - p_{11} &= 0, \\
p_9 + p_{10} - p_{13} - p_{14} &= 0, & p_5 + p_7 - p_{13} - p_{15} &= 0, \\
p_3 + p_4 - p_7 - p_8 &= 0, & p_{11} + p_{12} - p_{15} - p_{16} &= 0, \\
p_2 + p_4 - p_{10} - p_{12} &= 0, & p_6 + p_8 - p_{14} - p_{16} &= 0,
\end{aligned} \tag{45}$$

which is referred to as the *causal communication constraint* [26]. Notice that the constraints (45), of course, also hold when p_i are factorizable and are written in terms of r, s, r', s' as in (43). Essentially, these constraints state that, on measurement, Alice's probability of obtaining

particular outcome (+1 or -1), when she goes for S_1 or S_2 , is independent of how Bob sets up his apparatus (i.e. along S'_1 or along S'_2). The same applies to Bob i.e. on measurement his probability of obtaining a particular outcome (+1 or -1), when he goes for S'_1 or S'_2 , is independent of how Alice sets up her apparatus (i.e. along S_1 or along S_2). Other authors may like to call the constraints (45) with some different name, for example, Winsberg and Fine [22] have described them as the *locality constraint*.

Notice that because of normalization (41) half of the Eqs. (45) are redundant that makes eight among sixteen probabilities p_i to be independent. A convenient solution [26] of the system (41, 45), for which the set of variables:

$$v = \{p_2, p_3, p_6, p_7, p_{10}, p_{11}, p_{13}, p_{16}\}, \quad (46)$$

is expressed in terms of the remaining set of variables:

$$\mu = \{p_1, p_4, p_5, p_8, p_9, p_{12}, p_{14}, p_{15}\}, \quad (47)$$

is given as follows

$$\begin{aligned} p_2 &= (1 - p_1 - p_4 + p_5 - p_8 - p_9 + p_{12} + p_{14} - p_{15})/2, \\ p_3 &= (1 - p_1 - p_4 - p_5 + p_8 + p_9 - p_{12} - p_{14} + p_{15})/2, \\ p_6 &= (1 + p_1 - p_4 - p_5 - p_8 - p_9 + p_{12} + p_{14} - p_{15})/2, \\ p_7 &= (1 - p_1 + p_4 - p_5 - p_8 + p_9 - p_{12} - p_{14} + p_{15})/2, \\ p_{10} &= (1 - p_1 + p_4 + p_5 - p_8 - p_9 - p_{12} + p_{14} - p_{15})/2, \\ p_{11} &= (1 + p_1 - p_4 - p_5 + p_8 - p_9 - p_{12} - p_{14} + p_{15})/2, \\ p_{13} &= (1 - p_1 + p_4 + p_5 - p_8 + p_9 - p_{12} - p_{14} - p_{15})/2, \\ p_{16} &= (1 + p_1 - p_4 - p_5 + p_8 - p_9 + p_{12} - p_{14} - p_{15})/2. \end{aligned} \quad (48)$$

The relationships (48) between joint probabilities arise because both the normalization condition (41) and the causal communication constraint (45) are fulfilled.

From Eqs. (48) one can obtain other constraints considering that the sum of any combination of probabilities from the set v must be non-negative. In the following are some results to be used later in this paper. In (48) the sum $p_2 + p_7$ is non-negative and it requires that

$$p_1 + p_8 \leq 1. \quad (49)$$

In (48) the sum $p_3 + p_{10}$ is non-negative and it requires that

$$p_1 + p_{12} \leq 1. \quad (50)$$

Similarly, the sum $p_6 + p_{13}$ is non-negative and it requires that

$$p_8 + p_{13} \leq 1. \quad (51)$$

Both of the inequalities (49, 50) are found useful in developing a quantum version of PD. The inequality (51) is found useful in developing quantum version of SH.

Using (42) in (15), with the assumption that (x^*, y^*) is a NE, one obtains

$$\begin{aligned} \Pi_A(x^*, y^*) - \Pi_A(x, y^*) &= (x^* - x)[y^* \{K\Omega_1 + L\Omega_2 + M\Omega_3 + N\Omega_4\} + \\ &\quad \{K(p_5 - p_{13}) + L(p_6 - p_{14}) + M(p_7 - p_{15}) + N(p_8 - p_{16})\}] \geq 0, \end{aligned} \quad (52)$$

and

$$\begin{aligned} \Pi_B(x^*, y^*) - \Pi_B(x^*, y) &= (y^* - y)[x^* \{K\Omega_1 + M\Omega_2 + L\Omega_3 + N\Omega_4\} + \\ &\quad \{K(p_9 - p_{13}) + M(p_{10} - p_{14}) + L(p_{11} - p_{15}) + N(p_{12} - p_{16})\}] \geq 0, \end{aligned} \quad (53)$$

where

$$\begin{aligned} \Omega_1 &= p_1 - p_5 - p_9 + p_{13}, & \Omega_2 &= p_2 - p_6 - p_{10} + p_{14}, \\ \Omega_3 &= p_3 - p_7 - p_{11} + p_{15}, & \Omega_4 &= p_4 - p_8 - p_{12} + p_{16}. \end{aligned} \quad (54)$$

Now use (48) to write (52) and (53) in terms of the probabilities appearing in the set μ given in (47) to obtain

$$\begin{aligned} \Pi_A(x^*, y^*) - \Pi_A(x, y^*) &= (1/2)(x^* - x)[y^* \Delta_3 \times \\ &(1 + p_1 + p_4 - p_5 - p_8 - p_9 - p_{12} - p_{14} - p_{15}) - \\ &\{\Delta_3(1 - p_5 - p_8 - p_{14} - p_{15}) + \\ &(\Delta_1 + \Delta_2)(p_1 - p_4 - p_9 + p_{12})\}] \geq 0, \end{aligned} \quad (55)$$

$$\begin{aligned} \Pi_B(x^*, y^*) - \Pi_B(x^*, y) &= (1/2)(y^* - y)[x^* \Delta_3 \times \\ &(1 + p_1 + p_4 - p_5 - p_8 - p_9 - p_{12} - p_{14} - p_{15}) - \\ &\{\Delta_3(1 - p_9 - p_{12} - p_{14} - p_{15}) + \\ &(\Delta_1 + \Delta_2)(p_1 - p_4 - p_5 + p_8)\}] \geq 0. \end{aligned} \quad (56)$$

Notice that the probabilities associated to the EPR experiments can be factorized only for certain directions of measurements even for singlet states. For these directions the game played using EPR experiments can thus be interpreted within the four-coin setup.

Essentially, we obtain quantum game from the classical as follows. Referring to the four-coin setup developed in the Section 3, the factorizability of associated probabilities in terms of r, s, r', s' allows us to translate the requirement that the resulting game has a classical interpretation into certain constraints on r, s, r', s' . We find that from factorizability the relations (44) follow and from these relations the constraints on r, s, r', s' can be re-expressed in terms of p_i ($1 \leq i \leq 16$). We now obtain a quantum version of the game by retaining these constraints and afterwards allowing p_i to become non-factorizable. In this procedure retaining the constraints ensures that classical outcome results when probabilities become factorizable.

5.1 Examples

In the following we consider the impact of non-factorizable probabilities on the NE in PD, SH, and the Chicken game.

5.1.1 Prisoner's Dilemma

Recall that Section 3 states the result that when PD is played with four coins we require the condition (19) to hold if the strategy pair (S_2, S'_2) is to exist as a NE. Along with this the probabilities p_i are to be factorizable.

This motivates to construct a quantum version of PD when probabilities p_i are not factorizable while the constraint (19) remains valid. The condition (19) ensures that with factorizable probabilities the game can be interpreted classically.

Notice that when the probabilities p_i are factorizable, i.e. they can be written as in (43), the constraint (19) can hold when numerical values are assigned to certain probabilities among p_i :

$$p_5 = 0, p_7 = 0, p_9 = 0, p_{10} = 0, p_{16} = 1, \quad (57)$$

where, because of the normalization (41), $p_{16} = 1$ requires that $p_{13} = 0, p_{14} = 0$, and $p_{15} = 0$. This can also be noticed more directly from (44). This assignment of values to certain probabilities reduces Eqs. (41) and Eqs. (45) to

$$\begin{aligned} p_1 + p_2 + p_3 + p_4 &= 1, \\ p_1 + p_2 &= p_6, & p_1 + p_3 &= p_{11}, \\ p_3 + p_4 &= p_8, & p_{11} + p_{12} &= 1, \\ p_2 + p_4 &= p_{12}, & p_6 + p_8 &= 1. \end{aligned} \quad (58)$$

Substituting from (57) into (55, 56) gives

$$\begin{aligned} \Pi_A(x^*, y^*) - \Pi_A(x, y^*) &= (1/2)(x^* - x)[y^* \Delta_3(1 + p_1 + p_4 - p_8 - p_{12}) - \\ &\{\Delta_3(1 - p_8) + (\Delta_1 + \Delta_2)(p_1 - p_4 + p_{12})\}] \geq 0; \end{aligned} \quad (59)$$

$$\begin{aligned}
& \Pi_B(x^*, y^*) - \Pi_B(x^*, y) \\
&= (1/2)(y^* - y)[x^* \Delta_3(1 + p_1 + p_4 - p_8 - p_{12}) - \\
& \{ \Delta_3(1 - p_{12}) + (\Delta_1 + \Delta_2)(p_1 - p_4 + p_8) \}] \geq 0.
\end{aligned} \tag{60}$$

Note that from (58) we obtain $(1 - p_8) = p_6 = p_1 + p_2$, that for factorizable probabilities, becomes equal to r when we refer to Eqs. (44). Similarly, from (58) we obtain $(p_1 - p_4 + p_{12}) = p_1 + p_2 = r$. Substituting these in (59) along with that $x^* = 0 = y^*$ gives $\Pi_A(0, 0) - \Pi_A(x, 0) = xr\Delta_2 \geq 0$. In similar way we find from (58) that $(1 - p_{12}) = p_{11} = p_1 + p_3$ which for factorizable probabilities becomes equal to r' when we use Eqs. (44). Likewise, from (58) we obtain $(p_1 - p_4 + p_8) = p_1 + p_3 = r'$. Substituting these in (60), along with that $x^* = 0 = y^*$, gives $\Pi_B(0, 0) - \Pi_B(0, y) = yr'\Delta_2 \geq 0$. This can be described as follows: When probabilities p_1, p_4, p_8 and p_{12} are factorizable and the values assigned to them in (57) hold, the inequalities (59, 60) ensure that the strategy pair (S_2, S'_2) becomes a NE.

Now we ask about the fate of the NE strategy pair (S_2, S'_2) when in (57) the values assigned to certain probabilities, resulting from the requirement (19), hold while p_i do not remain factorizable in terms of r, s, r', s' . Allow the probabilities p_1, p_4, p_8 and p_{12} not to be factorizable and use (57) in (48) to get $1 - p_1 + p_4 - p_8 - p_{12} = 0$ and the inequalities (59, 60) take the form:

$$\begin{aligned}
& \Pi_A(x^*, y^*) - \Pi_A(x, y^*) \\
&= (x^* - x) [\Delta_2 \{y^* - (1 - p_8)/p_1\} - \Delta_1 y^*] p_1 \geq 0,
\end{aligned} \tag{61}$$

$$\begin{aligned}
& \Pi_B(x^*, y^*) - \Pi_B(x^*, y) \\
&= (y^* - y) [\Delta_2 \{x^* - (1 - p_{12})/p_1\} - \Delta_1 x^*] p_1 \geq 0,
\end{aligned} \tag{62}$$

where Δ_1 and Δ_2 are defined in the Section (3.2.1). Note that (49) and (50) give $1 \leq (1 - p_8)/p_1$ and $1 \leq (1 - p_{12})/p_1$ so that

$$\{y^* - (1 - p_8)/p_1\} \leq 0, \quad \{x^* - (1 - p_{12})/p_1\} \leq 0, \tag{63}$$

which results, once again, in the strategy pair $(x^*, y^*) = (0, 0)$ being a NE, which is the classical outcome of the game.

Notice that this NE emerges for non-factorizable EPR probabilities along with our requirement that factorizable probabilities must lead to the classical game. This result for PD appears to diverge away from the reported results in quantum games [2]. We believe that part of the reason resides with how payoff relations and players' strategies are defined in the present framework, which exploits EPR setup for playing a quantum game.

5.1.2 Stag Hunt

Section 3.2.2 describes playing SH in the four-coin setup for which three NE emerge. For each of these three NE there correspond constraints on r, s, r', s' for factorizable probabilities. In the following we first translate these constraints in terms of the EPR probabilities p_i and afterwards allow p_i to assume non-factorizable values when the constraints on r, s, r', s' , expressed in terms of p_i , continue to hold. In the following we follow this procedure for each individual NE that arises when SH is played in the four-coin setup.

$(x^*, y^*)_1 = (0, 0)$: Refer to Eqs. (23) in Section 3.2.2 and consider the NE $(x^*, y^*)_1 = (0, 0)$ for which the constraint on probabilities are (19) as it is the case with PD. Analysis for quantum PD from Section 5.1.1, therefore, remains valid and we can directly use the inequalities (61, 62, 63) to obtain

$$\begin{aligned}
& \Pi_A(x^*, y^*) - \Pi_A(x, y^*) = (x^* - x) \{y^* \Delta_3 - \Delta_2(1 - p_8)/p_1\} p_1 \geq 0, \\
& \Pi_B(x^*, y^*) - \Pi_B(x^*, y) = (y^* - y) \{x^* \Delta_3 - \Delta_2(1 - p_{12})/p_1\} p_1 \geq 0,
\end{aligned} \tag{64}$$

where $\Delta_3 > \Delta_2 > 0$. This gives rise to three equilibria:

$$\begin{aligned}
(x^*, y^*)_1^{Q_a} &= (0, 0); \\
(x^*)_2^{Q_a} &= (\Delta_2/\Delta_3)(1 - p_{12})/p_1, \quad (y^*)_2^{Q_a} = (\Delta_2/\Delta_3)(1 - p_8)/p_1; \\
(x^*, y^*)_3^{Q_a} &= (1, 1);
\end{aligned} \tag{65}$$

where the superscript Q refers to ‘quantum’. From the relations (64, 65) and the inequalities (49, 50) it turns out that $(x^*, y^*)_1^{Q_a}$ emerges without any further constraints apart from the ones given by re-expressed form of (19) i.e. (57); $\{(x^*)_2^{Q_a}, (y^*)_2^{Q_a}\}$ emerges when non-factorizable probabilities are such that, apart from (57) to hold, both $(\Delta_2/\Delta_3)(1 - p_{12})/p_1$ and $(\Delta_2/\Delta_3)(1 - p_8)/p_1$ have values in the interval $[0, 1]$; and $(x^*, y^*)_3^{Q_a}$ emerges when, apart from (57) being true, both $\{\Delta_3 - \Delta_2(1 - p_8)/p_1\}$ and $\{\Delta_3 - \Delta_2(1 - p_{12})/p_1\}$ are non negative.

$(x^*, y^*)_2 = (\Delta_2/\Delta_3, \Delta_2/\Delta_3)$: Refer to Section 3.2.2 and use (44) and (45), along with the normalization (41), to express the constraints (28) as

$$p_1 = 1 = p_6 \text{ and } p_{11} = 1 = p_{16}. \tag{66}$$

The normalization (41), then, assigns zero value to the remaining 12 probabilities. Now substitute the constraints (66) in Eqs. (55, 56) to obtain the NE inequalities that will correspond to the non-factorizable probabilities:

$$\begin{aligned}
\Pi_A(x^*, y^*) - \Pi_A(x, y^*) &= (x^* - x)[y^* \Delta_3 - \Delta_2] \geq 0, \\
\Pi_B(x^*, y^*) - \Pi_B(x^*, y) &= (y^* - y)[x^* \Delta_3 - \Delta_2] \geq 0.
\end{aligned} \tag{67}$$

This results in identical to the classical situation and the three NE:

$$\begin{aligned}
(x^*, y^*)_1^{Q_b} &= (0, 0); \\
(x^*, y^*)_2^{Q_b} &= (\Delta_2/\Delta_3, \Delta_2/\Delta_3); \\
(x^*, y^*)_3^{Q_b} &= (1, 1);
\end{aligned} \tag{68}$$

emerge when (66) hold.

$(x^*, y^*)_3 = (1, 1)$: For this NE in (23) the constraint on probabilities are (27) i.e. $r = 0 = r'$. For factorizable probabilities, this constraint can be rewritten using normalization (41) along with (44, 45) as

$$\begin{aligned}
p_5 &= 0, \quad p_6 = 0, \quad p_9 = 0, \quad p_{11} = 0, \\
p_7 + p_8 &= 1 = p_{10} + p_{12}, \quad p_4 = 1,
\end{aligned} \tag{69}$$

from which using the normalization (41) it then follows that $p_1 = 0, p_2 = 0, p_3 = 0$. The constraints (69) reduce the Nash inequalities (55, 56) to

$$\begin{aligned}
&\Pi_A(x^*, y^*) - \Pi_A(x, y^*) \\
&= (1/2)(x^* - x)[y^* \Delta_3(2 - p_8 - p_{12} - p_{14} - p_{15}) - \\
&\quad \{\Delta_3(1 - p_8 - p_{14} - p_{15}) + (\Delta_1 + \Delta_2)(-1 + p_{12})\}] \geq 0;
\end{aligned} \tag{70}$$

$$\begin{aligned}
&\Pi_B(x^*, y^*) - \Pi_B(x^*, y) \\
&= (1/2)(y^* - y)[x^* \Delta_3(2 - p_8 - p_{12} - p_{14} - p_{15}) - \\
&\quad \{\Delta_3(1 - p_{12} - p_{14} - p_{15}) + (\Delta_1 + \Delta_2)(-1 + p_8)\}] \geq 0.
\end{aligned} \tag{71}$$

Using the constraints (69) in the 7-th Equation in (48) results in $p_{13} = (2 - p_8 - p_{12} - p_{14} - p_{15})/2$ which then simplifies the Nash inequalities (70, 71) to

$$\begin{aligned}
\Pi_A(x^*, y^*) - \Pi_A(x, y^*) &= (x^* - x) \{-(1 - y^*)p_{13}\Delta_3 + (1 - p_{12})\Delta_2\} \geq 0, \\
\Pi_B(x^*, y^*) - \Pi_B(x^*, y) &= (y^* - y) \{-(1 - x^*)p_{13}\Delta_3 + (1 - p_8 - p_{13})\Delta_2\} \geq 0,
\end{aligned} \tag{72}$$

and gives rise to three NE that are described below.

$(x^*, y^*)_1^{Q_c}$: For this NE we have $(x^*, y^*)_1^{Q_c} = (0, 0)$ and the inequalities (72) become

$$\begin{aligned}\Pi_A(0, 0) - \Pi_A(x, 0) &= xp_{13}\Delta_3 \{1 - (\Delta_2/\Delta_3)(1 - p_{12})/p_{13}\} \geq 0, \\ \Pi_B(0, 0) - \Pi_B(0, y) &= yp_{13}\Delta_3 \{1 - (\Delta_2/\Delta_3)(1 - p_8 - p_{13})/p_{13}\} \geq 0,\end{aligned}\quad (73)$$

where from (50, 51) we have $(1 - p_{12}) \geq 0$ and $(1 - p_8 - p_{13}) \geq 0$. That is, $(x^*, y^*)_1^{Q_c} = (0, 0)$ will be a NE when p_8, p_{12} , and p_{13} are such that

$$1 \geq (\Delta_2/\Delta_3)(1 - p_{12})/p_{13}, \quad 1 \geq (\Delta_2/\Delta_3)(1 - p_8 - p_{13})/p_{13}, \quad (74)$$

hold true along with the constraints (69).

$(x^*, y^*)_2^{Q_c}$: From the inequalities (72) the strategy pair $(x^*, y^*)_2^{Q_c} = (x^*, y^*)$ where the strategy pair $x^* = 1 - (\Delta_2/\Delta_3)(1 - p_8 - p_{13})/p_{13}$, $y^* = 1 - (\Delta_2/\Delta_3)(1 - p_{12})/p_{13}$ can exist as a NE when p_8, p_{12} , and p_{13} are such that $x^*, y^* \in [0, 1]$. Together with this the constraints (69) are to hold.

$(x^*, y^*)_3^{Q_c}$: For the possible NE $(x^*, y^*)_3^{Q_c} = (1, 1)$ the inequalities (72) become

$$\begin{aligned}\Pi_A(x^*, y^*) - \Pi_A(x, y^*) &= (1 - x)(1 - p_{12})\Delta_2 \geq 0, \\ \Pi_B(x^*, y^*) - \Pi_B(x^*, y) &= (1 - y)(1 - p_8 - p_{13})\Delta_2 \geq 0,\end{aligned}\quad (75)$$

which are to hold along with that the constraints (69) being true. Using $\Delta_1, \Delta_1 > 0$ with (50, 51) we find that the inequalities (75) will always hold and the only requirement for the strategy pair $(x^*, y^*)_3^{Q_c} = (1, 1)$ to be a NE is that the constraints (69) hold.

So that the list of possible NE, that can arise in the quantum SH, consists of five members i.e. $(0, 0)$, $(1, 1)$, $\{\Delta_2(1 - p_{12})/\Delta_3 p_1, \Delta_2(1 - p_8)/\Delta_3 p_1\}$, $(\Delta_2/\Delta_3, \Delta_2/\Delta_3)$, and $\{[1 - \Delta_2(1 - p_8 - p_{13})/\Delta_3 p_{13}], [1 - \Delta_2(1 - p_{12})/\Delta_3 p_{13}]\}$. Which one, or more, from this list are going to arise depends on the set of non-factorizable probabilities. For example, we notice that there exist [26] two sets of non-factorizable probabilities that maximally violate the quantum prediction of the Clauser-Holt-Shimony-Horne (CHSH) sum of correlations. The first set is

$$\begin{aligned}p_j &= (2 + \sqrt{2})/8 \text{ for all } p_j \in \mu, \\ p_k &= (2 - \sqrt{2})/8 \text{ for all } p_k \in \nu,\end{aligned}\quad (76)$$

and the second set is

$$\begin{aligned}p_j &= (2 - \sqrt{2})/8 \text{ for all } p_j \in \mu, \\ p_k &= (2 + \sqrt{2})/8 \text{ for all } p_k \in \nu,\end{aligned}\quad (77)$$

where μ and ν are defined in (46) and in (47), respectively. The probabilities in these sets are non-factorizable because for both sets a solution of the Eqs. (43) will involve one or more of the probabilities r, s, r', s' being negative or greater than one. Now for SH the requirement that factorizable probabilities are to lead to classical game gives rise to three sets of constraints on EPR probabilities given by (57), (66), and (69). These sets of constraints correspond to the NE $(x^*, y^*)_1 = (0, 0)$, $(x^*, y^*)_2 = (\Delta_2/\Delta_3, \Delta_2/\Delta_3)$, and $(x^*, y^*)_3 = (1, 1)$ respectively. Unfortunately, the probabilities from either of the two sets (76, 77), which maximally violate the quantum prediction of the CHSH sum of correlations, do not satisfy these constraints. Stated otherwise, the probabilities from the sets (76, 77) are in conflict with the requirement that factorizable probabilities must lead to the classical game of SH. However, other sets of non-factorizable probabilities can be found that are consistent with this requirement and, depending on the elements of a set, one or more out of five possible NE may emerge. This situation can be described by saying that in SH non-factorizability can lead to new NE but, unfortunately, either of the sets (76, 77) cannot be used for this purpose.

5.1.3 Chicken game

Refer to Section 3.2.3 and use Eqs. (36) express the constraints on r, s, r', s' in this setup. These constraints are imposed for the strategy pair $x^* = \alpha/(\alpha + \beta) = y^*$ is to be a NE. Use Eqs. (44) and the normalization (41) to translate these constraints in terms of the EPR probabilities p_i :

$$\alpha(p_2 + p_4) = \beta(p_5 + p_7), \quad (78)$$

$$\alpha(p_3 + p_4) = \beta(p_9 + p_{10}). \quad (79)$$

Addition and subtraction of (79) to and from (78) gives

$$\begin{aligned} \alpha(p_2 + p_3 + 2p_4) &= \beta(p_5 + p_7 + p_9 + p_{10}), \\ \alpha(p_2 - p_3) &= \beta(p_5 + p_7 - p_9 - p_{10}), \end{aligned} \quad (80)$$

and Eqs. (48) then allow us to re-express Eqs. (80) in terms of the probabilities in set μ , defined in (47), to obtain

$$\begin{aligned} (\alpha/\beta - 1)(1 - p_1 + p_4) &= p_5 - p_8 + p_9 - p_{12}, \\ (1 + \beta/\alpha)(-p_{14} + p_{15}) &= p_5 - p_8 - p_9 + p_{12}. \end{aligned} \quad (81)$$

Two probabilities can be eliminated from the inequalities (55, 56) using Eqs. (81). We select (arbitrarily) these to be p_{12} and p_{15} and express them in terms of other probabilities in the set μ i.e.

$$\begin{aligned} p_{12} &= p_5 - p_8 + p_9 - (\alpha/\beta - 1)(1 - p_1 + p_4), \\ p_{15} &= p_{14} + \frac{2(p_5 - p_8) - (\alpha/\beta - 1)(1 - p_1 + p_4)}{1 + \beta/\alpha}. \end{aligned} \quad (82)$$

Notice that for the Chicken game, defined in (29), the definition of $\triangle_{1,2}$ in Section 3.2.1 gives

$$\triangle_1 = \beta, \quad \triangle_2 = -\alpha. \quad (83)$$

Now eliminate p_{12} and p_{15} from the inequalities (55, 56) using Eqs. (82) and substitute from Eqs. (83). The inequalities (55, 56) then read

$$\begin{aligned} \Pi_A(x^*, y^*) - \Pi_A(x, y^*) &= \alpha(x^* - x)[-y^*\{(p_1 - 2p_5 - p_9 - p_{14} + p_8) + \\ &\quad (1 - p_1 + p_4)\alpha/\beta + (p_1 - p_5 - p_9 - p_{14})\beta/\alpha\} + \\ &\quad \{(1 - p_5 - p_{14}) - (p_5 + p_{14})\beta/\alpha\}]; \end{aligned} \quad (84)$$

$$\begin{aligned} \Pi_B(x^*, y^*) - \Pi_B(x^*, y) &= \alpha(y^* - y)[-x^*\{(p_1 - 2p_5 - p_9 - p_{14} + p_8) + \\ &\quad (1 - p_1 + p_4)\alpha/\beta + (p_1 - p_5 - p_9 - p_{14})\beta/\alpha\} + \\ &\quad \{(p_1 - p_4 - 2p_5 + 2p_8 - p_9 - p_{14}) + \\ &\quad (1 - p_1 + p_4)\alpha/\beta - (p_9 + p_{14})\beta/\alpha\}]. \end{aligned} \quad (85)$$

Inequalities (84, 85) ensure that for factorizable probabilities the classical NE $x^* = \alpha/(\alpha + \beta) = y^*$ comes out as the outcome of the game. What is the fate of this equilibrium when probabilities are not factorizable? To address this question we consider a special case when $\alpha = \beta$, for which $x^* = 1/2 = y^*$ is the classical mixed-strategy outcome of the game. To obtain this outcome within the four-coin setup the constraints on r, s, r', s' are given in Eqs. (38). The inequalities (84, 85) reduce to

$$\begin{aligned} \Pi_A(x^*, y^*) - \Pi_A(x, y^*) &= \alpha(x^* - x)\{-y^*(1 + p_1 + p_4 - 3p_5 + p_8 - 2p_9 - 2p_{14}) + \\ &\quad (1 - 2p_5 - 2p_{14})\}; \end{aligned} \quad (86)$$

$$\begin{aligned}
& \Pi_B(x^*, y^*) - \Pi_B(x^*, y) \\
&= \alpha(y^* - y)\{-x^*(1 + p_1 + p_4 - 3p_5 + p_8 - 2p_9 - 2p_{14}) + \\
& (1 - 2p_5 + 2p_8 - 2p_9 - 2p_{14})\}.
\end{aligned} \tag{87}$$

Notice that the inequalities (72) do not allow either of the strategy pairs $(x^*, y^*) = (1, 0)$ and $(x^*, y^*) = (0, 1)$ to be NE. Like it has been the case with quantum SH, we now ask which of these nine possible NE will emerge when probabilities become non-factorizable. To answer this we refer to the set (76) of probabilities and assign the value $(2 + \sqrt{2})/8$ to each of the probabilities $p_1, p_4, p_5, p_8, p_9, p_{14}$. Using Eqs. (82) the assumption that $\alpha = \beta$ then also assigns the same value, i.e. $(2 + \sqrt{2})/8$, to both p_{12} and p_{15} . The inequalities (86, 87) are

$$\begin{aligned}
\Pi_A(x^*, y^*) - \Pi_A(x, y^*) &= (\alpha/\sqrt{2})(x^* - x)(y^* - 1) \geq 0, \\
\Pi_B(x^*, y^*) - \Pi_B(x^*, y) &= (\alpha/\sqrt{2})(y^* - y)(x^* - 1) \geq 0,
\end{aligned} \tag{88}$$

with the result that the strategy pairs $(x^*, y^*) = (1, 1), (0, 0)$ emerge as the new equilibria³. So that, in this case, the set (76) of non-factorizable probabilities indeed leads to the new equilibria of the game. Using (15) one finds that at the equilibrium $(1, 1)$ both players get $\alpha(2 - \sqrt{2})/4$ while at $(0, 0)$ both players get $\alpha(2 + \sqrt{2})/4$ as their payoffs.

Note that, when re-expressed in terms of the EPR probabilities p_i using Eqs. (44), the constraints (38) can be written as

$$p_1 + p_2 + p_9 + p_{10} = 1 = p_1 + p_3 + p_5 + p_7, \tag{89}$$

which, of course, continue to hold for the set (76) when the probabilities p_i are allowed to be non-factorizable.

Similarly, referring to the second set (77), we assign the value $(2 - \sqrt{2})/8$ to the probabilities $p_1, p_4, p_5, p_8, p_9, p_{14}$ that appear in (86) and (87). The Eqs. (82), with the assumption that $\alpha = \beta$, then assign the value of $(2 - \sqrt{2})/8$ both to p_{12} and p_{15} and the inequalities (86, 87) read

$$\begin{aligned}
\Pi_A(x^*, y^*) - \Pi_A(x, y^*) &= -(\alpha/\sqrt{2})(x^* - x)(y^* - 1) \geq 0, \\
\Pi_B(x^*, y^*) - \Pi_B(x^*, y) &= -(\alpha/\sqrt{2})(y^* - y)(x^* - 1) \geq 0,
\end{aligned} \tag{90}$$

which are the same as the ones given in (88), apart from extra negative signs. This results in three strategy pairs $(x^*, y^*) = (1, 1), (1, 0), (0, 1)$ to come out as the equilibria. Once again, using (15) one finds that at all of these three equilibria both players get equally rewarded by the amount $\alpha(2 + \sqrt{2})/4$. Hence, while referring to (31), we find that in this special case when $\alpha = \beta$ the set (77) of probabilities leads to new equilibria of the game. Notice that, like it is the case with the set (76) of probabilities, the constraints (89) continue to hold also for the set (77) when p_i are allowed to be non-factorizable.

6 Discussion

In typical quantization procedure [2, 5] of a two-player game, two quantum bits (qubits) are in a quantum correlated (entangled) state that are given to the players Alice and Bob. Players' strategies consist of performing unitary actions on their respective qubits. Classical game remains a subset of the quantum game in that both players can play quantum strategies that correspond to the strategies available classically.

As more choices are allowed to the players that now also include superpositions of their classical moves, it gives ground to the argument⁴ that Enk and Pike [27] have put forward. The setup

³Referring to (31) we recall that when $\alpha = \beta$ there are three equilibria i.e. $(1, 0), (0, 1)$ and $(1/2, 1/2)$ in the classical Chicken game, at which they get rewarded by $(\alpha, \beta), (\beta, \alpha)$ and $(\alpha\beta/(\alpha + \beta), \alpha\beta/(\alpha + \beta))$, respectively. Here the first and the second entry refers to Alice's and Bob's reward, respectively.

⁴Enk and Pike [27] argument can be described as follows. The extended set of players' moves allows us to construct an extended payoff matrix that includes extra available moves. Enk and Pike interpret this by saying that the 'essence' of a quantized game can be captured by a different classical game and it is the new game that is constructed and solved and *not* the original classical game.

proposed in this paper uses EPR experiments to play a two-player game and a quantum game is associated to a classical game such that it becomes hard to construct an Enk and Pike-type argument as both the payoff relations and the players' sets of strategies remain identical [28] in the classical and the associated quantum game.

In the present setup it is non-factorizability – responsible for the violation of Bell inequality in EPR experiments – that gives rise to the new solutions in quantum game. When players play a game using a physical system for which joint probabilities are factorizable the classical game results always. In other words, the role of non-factorizable probabilities is sought in the game-theoretic solution concept of a NE, when the physical realization for these probabilities is provided by the EPR experiments. This analysis introduces a new viewpoint in the area of quantum games in which non-factorizability gets translated into the language of game theory.

The argument put forward in this paper can be described as follows. Firstly, players' payoffs are re-expressed in the form $\Pi_{A,B}(p_i, x, y, \mathcal{A}, \mathcal{B})$ where p_i are the sixteen joint probabilities; x, y are players strategies, and \mathcal{A}, \mathcal{B} are players' payoff matrices defined in (9). Secondly, Nash inequalities are used to impose constraints on p_i that ensure that with factorizable p_i the game has classical outcome and the resulting payoffs can be interpreted in terms of classical mixed-strategy game. It is achieved by playing the game in the four-coin setup and using Nash inequalities to obtain constraints on the coin probabilities r, s, r', s' which reproduce the outcome of the classical mixed-strategy game. Using (44), which results from factorizability, these constraints on r, s, r', s' are then translated in terms of constraints on p_i . Thirdly, while referring to the EPR setup, p_i are allowed to be non-factorizable when the constraints on p_i continue to hold. Fourthly, and lastly, it is observed how non-factorizability leads to the emergence of new solutions of the game.

Note that for a game different sets of constraints are defined depending on which NE is to be the solution of the game. For example, for three NE in Chicken we require three different sets of constraints on r, s, r', s' . Considering one of these three sets at a time we repeat the four steps stated above. The same procedure is then repeated for other sets of constraints corresponding to other NE.

That is, in this setup not all solutions of a game are re-expressed in terms of a single set of constraints on r, s, r', s' . Instead, a separate set of constraints is found for each NE. It seems that the four-coin setup is the minimal arrangement that allows one to introduce, in a smooth way, the EPR probabilities into a game-like setting. We suggest that with increasing the number of coins, shared by each of the two players, all the NE of a game can be translated to a single constraint on the underlying coin probabilities which are subsequently translated in terms of p_i . This will then allow to see the role of non-factorizability on solution of a game from a single set of constraints. However, this will be obtained at a price: Firstly, more coins will be involved resulting in more mathematical complexity; secondly, for more coins player's strategy will need to be redefined such that it permits to incorporate EPR probabilities.

Note that the usual approach uses entangled states to construct quantum games and this paper uses non-factorizability to the same end. Mathematically, non-factorizability comes out to be a stronger condition than the condition that translates entanglement into constraints on joint probabilities. That is, a non-factorizable set of probabilities always corresponds to some entangled state but an entangled state can produce a factorizable set of probabilities. For example, in case of singlet state the outcomes of two measurements violate Bell inequality only along certain directions, and not along other directions. In other words, in a quantum game exploiting entangled states, the joint probabilities may still be factorizable but for a quantum game, resulting from non-factorizable probabilities, Bell inequality is bound to be violated. Non-factorizability being a stronger condition may well be suggested as the reason why it cannot be helpful to escape from the classical outcome in PD.

The proposed setup demonstrates how non-factorizability can change outcome of a game. We suggest to extend [29] this setup to analyze multi-player quantum games [3] where players share physical systems for which joint probabilities cannot be factorized.

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